Chamber Pressure and Thrust in Liquid Engine Derived from First Principles

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March 20, 2020

$\dot{m}$= mass flow rate
p_0= chamber pressure
p_{ss}= steady-state chamber pressure
ρ_0= density of gas in combustion chamber
T_0= temperature in combustion chamber
V_0= free volume in combustion chamber
R= specific gas constant of gas
γ= ratio of specific heat of gas
c^*= characteristic velocity
C_f= thrust coefficient
τ= thrust
τ_{ss}= steady-state thrust

*Assumption: Flow is already choked through the nozzle

First we will start by coming up with an equation for the mass accumulation rate in the combustion chamber.

$$\frac{dM}{dt} = \frac{d}{dt}(\rho_0 V_0) = V_0 \frac{d\rho_0}{dt} \tag{1}$$

V_0 can be taken out of the derivative, since the free volume of the combustion chamber is unchanging.

Now we can do a mass balance

$$\dot{m_{entering}} = \dot{m_{exiting}} + \frac{dM}{dt} \tag{2}$$

We can break the mass entering into the mass of oxidizer and fuel. Also if we assume that flow is choked, we then get

$$(\dot{m}_o + \dot{m}_f) = V_0 \frac{d\rho_0}{dt} + p_0 A^* \sqrt{\left[\frac{\gamma}{RT_0}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]} \tag{3}$$

The ideal gas law is

$$p_0 = \rho_0 R T_0 \tag{4}$$

Since the chamber temperature is essentially constant during the burn, we can differentiate the expression and rearrange to get

$$\frac{d\rho_0}{dt} = \frac{1}{RT_0}\frac{dp_0}{dt} \tag{5}$$

This can be substituted into Equation 3 to get

$$(\dot{m}_o + \dot{m}_f) = \frac{V_0}{RT_0}\frac{dp_0}{dt} + p_0 A^* \sqrt{\left[\frac{\gamma}{RT_0}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]} \tag{6}$$

Equation 6 governs the pressure in the combustion chamber as a function of time.

Steady state chamber pressure is achieved when $\frac{dp_0}{dt} = 0$. So we can show that steady-state chamber pressure is

$$p_{ss} = \frac{(\dot{m}_o + \dot{m}_f)}{A^*}\left[\frac{\gamma}{RT_0}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\right]^{-\frac{1}{2}} \tag{7}$$

The second term can be recognized to be c^*, so the formula can be rearranged to be:

$$p_{ss} = \frac{(\dot{m}_o + \dot{m}_f)c^*}{A^*} \tag{8}$$

And via the following thrust equation:

$$\tau = C_f A^* p_0 \tag{9}$$

It can be seen that steady-state thrust is given by

$$\tau_{ss} = C_f(\dot{m}_o + \dot{m}_f)c^* \tag{10}$$

The equation for thrust is a product of three terms. The first term represents how effective the nozzle is. The second term is is how much fuel and oxidizer is injected into the engine per second. The final term is a function of propellant chemistry and is a measure of combustion efficiency. To manipulate the thrust of a liquid engine, these are the three things you can change.

In liquid engines around reasonable sizes (4"x12"), steady-state pressure and thrust are achieved almost instantly after choked flow is achieved, so these formulas can be used to give you a very, very good estimate for the pressure and thrust of a liquid engine.